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ON THE MOTION OF A HOLLOW BODY FILLED WITH VISCOUS LIQUID ABOUT ITS CENTER OF MASS IN A POTENTIAL BODY-FORCE FIELD

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We consider the motion of a hollow solid body whose cavity is completely filled with a viscous liquid, assuming that the product of the Reynolds and Strouhal characteristic numbers for the flow of the viscous fluid in the cavity is small. We then show that the problem can be handled by methods used to investigate systems with a small parameter accompanying the higher derivatives and develop an algorithm for constructing an asymptotic expansion of the corresponding simultaneous system of Navier-Stokes and ordinary differential equations. Chernous'ko [1] constructed a system of ordinary differential equations which approximates the motion of the system in question outside the initial time integral, when the flow in the cavity is essentially unsteady. Our approach enables us to construct a solution without recourse to the additional conditions imposed by Chernous'ko on the derivatives of the angular velocities of the body, to evaluate the time-dependent "thickness" of the boundary layer, and to write out initial conditions for the system of equations proposed by Chernous'ko.

1. The basic equations of the problem. The motion of a hollow body completely filled with viscous liquid about its center of mass in a potential body-force field is described by the following system of equations in the coordinate system rigidly attached to the solid body [1]:

$$\frac{1}{\nu}\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\nabla)\,\mathbf{u}+\frac{d\boldsymbol{\omega}}{dt}\times\mathbf{r}+2\boldsymbol{\omega}\times\mathbf{u}\right]-\Delta\mathbf{u}+\nabla q=0, \qquad \operatorname{div}\mathbf{u}=0$$

$$J\,\frac{d\boldsymbol{\omega}}{dt}+\frac{d\mathbf{K}}{dt}+\boldsymbol{\omega}\times[J\boldsymbol{\omega}+\mathbf{K}]-\mathbf{M}=0$$

$$\mathbf{K}=\rho\int\limits_D(\mathbf{r}\times\mathbf{u})\,dm, \qquad q=\frac{1}{\nu}\left[\frac{p}{\rho}+U-{}^1/_2\,(\boldsymbol{\omega}\times\mathbf{r})^2\right] \tag{1.1}$$

$$\mathbf{u}\,|_\Gamma=0, \qquad \mathbf{u}\,(\mathbf{r},\,0)=\mathbf{u}^\circ(\mathbf{r}), \qquad \boldsymbol{\omega}\,(0)=\boldsymbol{\omega}^\circ$$

Here $\mathbf{u}$ is the relative velocity of an arbitrary point of the system; $\boldsymbol{\omega}$ is the absolute angular velocity of the body; $\mathbf{r}$ is the radius vector of the given point in the attached coordinate system; $J=J_0+J_1$, where J_0 is the inertia tensor of the body and J_1 is the inertia tensor of liquid "solidified" in the cavity; ν is the kinematic viscosity of the liquid; ρ is the density of the liquid; p is the pressure in the liquid; U is the potential of the external body forces; $\mathbf{M}$ is the moment of external forces; D is the cavity occupied by the liquid; Γ is the boundary of this cavity.

Investigation of Eqs. (1.1) consists in the simultaneous consideration of the boundary value problem for the general system of Navier-Stokes equations and the Cauchy problem for a system of ordinary differential equations.

In [1] it is shown that unsteady boundary value problems for the system of Navier-Stokes equations are solvable in the generalized sense at all instants provided the external forces are potential and the Reynolds number at the initial instant is small.

In solving the Cauchy problem for the above regular [3] systems of ordinary differential equations we must bear in mind the continuous dependence of the solutions on variations of the initial data and of the right sides of the equations. The continuous dependence of the solutions $\mathbf{u}\,(\mathbf{r},\,t,\,1\,/\,\nu)$ on the initial conditions $\mathbf{u}\,(\mathbf{r},\,0)$ and on the parameter $1\,/\,\nu$ also holds over a finite time interval for the regular system of Navier-Stokes equations (e.g. see [2]).

Let T_0 be the characteristic time of motion of the body relative to the center of mass and let L_0 be the characteristic linear size of the domain D. We introduce the parameter μ which is equal to the product of the Reynolds and Strouhal numbers for the flow of the viscous liquid in the cavity D,

$$\mu=\nu^{-1}T_0^{-1}L_0^{2} \tag{1.2}$$

Let us set $L_0=1,\ T_0=1$. Hence, $\mu=1\,/\,\nu$. We assume from now on that the parameter μ is small. In other words, we propose to construct an approximate solution of system (1.1), rejecting quantities of a certain order of smallness relative to μ.

2. Constructing the asymptotic expansion of the solution. Let us construct the asymptotic form of the solution of system (1.1) in the small parameter μ according to the procedure proposed in [3].

We formally construct the solution of system (1.1) in the form of series in powers of μ,

$$\begin{aligned} \mathbf{u}(\mathbf{r}, t, \mu) &= \mathbf{u}_0^1(\mathbf{r}, t) + \mu \mathbf{u}_1^1(\mathbf{r}, t) + \ldots \\ q(\mathbf{r}, t, \mu) &= q_0^1(\mathbf{r}, t) + \mu q_1^1(\mathbf{r}, t) + \ldots \\ \omega(t, \mu) &= \omega_0^1(t) + \mu \omega_1^1(t) + \ldots \end{aligned} \tag{2.1}$$

The initial conditions for the corresponding systems of equations which we obtain on substituting these series into system (1.1) will be specified below in a certain special way. For the present we merely note that determination of the functions $\mathbf{u}_k^1(\mathbf{r}, t), (k = 0, 1, \ldots)$ requires the solution of steady boundary value problems. This means that initial values must be specified for the functions $\omega_k^1(t)$ only.

Now let us formally construct the solution of system (1.1) (having first made the substitution of variables $\tau = t / \mu$) in the form of series,

$$\begin{aligned} \mathbf{u}(\mathbf{r}, \tau, \mu) &= \mathbf{u}_0^2(\mathbf{r}, \tau) + \mu \mathbf{u}_1^2(\mathbf{r}, \tau) + \ldots \\ q(\mathbf{r}, \tau, \mu) &= q_0^2(\mathbf{r}, \tau) + \mu q_1^2(\mathbf{r}, \tau) + \ldots \\ \omega(\tau, \mu) &= \omega_0^2(\tau) + \mu \omega_1^2(\tau) \end{aligned} \tag{2.2}$$

The initial conditions for the corresponding systems of equations in variations can be written as follows:

$$\mathbf{u}_0^2(\mathbf{r}, 0) = \mathbf{u}^\circ(\mathbf{r}), \quad \mathbf{u}_k^2(\mathbf{r}, 0) = 0, \quad \omega_0^2(0) = \omega^\circ, \quad \omega_k^2(0) = 0 \quad (k > 0) \tag{2.3}$$

Next we expand all the coefficients of series (2.1) in powers of t,

$$\begin{aligned} \mathbf{u}_k^1(\mathbf{r}, t) &= \mathbf{u}_{k0}^1(\mathbf{r}) + t \mathbf{u}_{k1}^1(\mathbf{r}) + \ldots \\ q_k^1(\mathbf{r}, t) &= q_{k0}^1(\mathbf{r}) + t q_{k1}^1(\mathbf{r}) + \ldots \\ \omega_k^1(t) &= \omega_{k0}^1 + t \omega_{k1}^1 + \ldots \end{aligned} \tag{2.4}$$

Let us substitute series (2.4) into (2.1), make the substitution $\tau = t / \mu$ in the resulting formal expansions, and regroup the terms of these expansions in such a way as to obtain series in powers of μ,

$$\begin{aligned} \mathbf{u}(\mathbf{r}, \tau, \mu) &= \mathbf{u}_{00}^1(\mathbf{r}) + \mu [\mathbf{u}_{10}^1(\mathbf{r}) + \tau \mathbf{u}_{01}^1(\mathbf{r})] + \cdots, \\ q(\mathbf{r}, \tau, \mu) &= q_{00}^1(\mathbf{r}) + \mu [q_{10}^1(\mathbf{r}) + \tau q_{01}^1(\mathbf{r})] + \cdots \\ \omega(\tau, \mu) &= \omega_{00}^1 + \mu [\omega_{10}^1 + \tau \omega_{01}^1] + \cdots \end{aligned} \tag{2.5}$$

Let us denote the coefficients of the kth power of μ in the resulting series (2.5) by $\mathbf{u}_k^3(\mathbf{r}, \tau)$, $q_k^3(\mathbf{r}, \tau)$, $\omega_k^3(\tau)$, respectively. Hence,

$$\begin{aligned} \mathbf{u}(\mathbf{r}, \tau, \mu) &= \mathbf{u}_0^3(\mathbf{r}, \tau) + \mu \mathbf{u}_1^3(\mathbf{r}, \tau) + \cdots \\ q(\mathbf{r}, \tau, \mu) &= q_0^3(\mathbf{r}, \tau) + \mu q_1^3(\mathbf{r}, \tau) + \cdots \\ \omega(\tau, \mu) &= \omega_0^3(\tau) + \mu \omega_1^3(\tau) + \cdots \end{aligned} \tag{2.6}$$

Finally, we construct the following expressions:

$$\begin{aligned} G_n(\mathbf{u}) = (\mathbf{u})_n^1 + (\mathbf{u})_n^2 - (\mathbf{u})_n^3, \quad G_n(q) = (q)_n^1 + (q)_n^2 - (q)_n^3 \\ G_n(\omega) = (\omega)_n^1 + (\omega)_n^2 - (\omega)_n^3 \end{aligned} \tag{2.7}$$

Here $(\cdot)_n^i$ $(i = 1, 2, 3)$ represents the partial sums of series (2.1), (2.2), (2.6). Expressions (2.7) are the partial sums of the asymptotic expansions of the solution of problem (1.1). This can be formulated more precisely as follows.

The inequalities

$$\|\mathbf{u}(\mathbf{r}, t, \mu) - G_n(\mathbf{u})\| < a\mu^{n+1}, \quad |\boldsymbol{\omega}(t, \mu) - G_n(\boldsymbol{\omega})| < a\mu^{n+1} \tag{2.8}$$

are valid for the solution of system (1.1) for sufficiently small μ and $t \in [0, T]$. Here a is a constant which does not depend on t and μ and $\|\cdot\|$ is the norm in $L_2(D)$. Let us prove this statement.

We begin by investigating the so-called "adjoint" system of equations (the rapid-motion system) relative to the initial point. We construct this system by setting first $\tau = t / \mu$ and then $\mu = 0$ in (1.1). This yields

$$\frac{\partial \mathbf{u}}{\partial \tau} + \frac{d\boldsymbol{\omega}}{d\tau} \times \mathbf{r} - \Delta \mathbf{u} + \nabla q, \quad \operatorname{div} \mathbf{u} = 0$$

$$J\frac{d\boldsymbol{\omega}}{d\tau} + \frac{d\mathbf{K}}{d\tau} = 0, \quad \mathbf{K} = \rho \int_D (\mathbf{r} \times \mathbf{u})\, dm \tag{2.9}$$

$$\mathbf{u}|_\Gamma = 0, \quad \mathbf{u}(\mathbf{r}, 0) = \mathbf{u}^\circ(\mathbf{r}), \quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}^\circ$$

Let us consider the behavior of the solutions of linear unsteady problem (2.9) as $\tau \to \infty$. From the last equation of system (2.9) we infer that

$$\boldsymbol{\omega}(\tau) = \boldsymbol{\omega}^\circ + \rho J^{-1} \int_D [\mathbf{r} \times (\mathbf{u}^\circ - \mathbf{u})]\, dm \tag{2.10}$$

We note, moreover, that a closed subsystem of partial differential equations for the variables $\mathbf{u}(\mathbf{r}, \tau)$, $q(\mathbf{r}, \tau)$ can be isolated from system (2.9),

$$\frac{\partial \mathbf{u}}{\partial \tau} - \rho J^{-1} \int_D \left(\mathbf{r} \times \frac{\partial \mathbf{u}}{\partial \tau}\right) dm \times \mathbf{r} - \Delta \mathbf{u} + \nabla q = 0, \quad \operatorname{div} \mathbf{u} = 0 \tag{2.11}$$

$$\mathbf{u}|_\Gamma = 0, \quad \mathbf{u}(\mathbf{r}, 0) = \mathbf{u}^\circ(\mathbf{r})$$

Let us set

$$\mathbf{u}(\mathbf{r}, \tau) = \mathbf{w}(\mathbf{r})\, e^{\lambda\tau}, \qquad q(\mathbf{r}, \tau) = s(\mathbf{r})\, e^{\lambda\tau} \tag{2.12}$$

in system (2.11).

This yields

$$\Delta \mathbf{w} - \nabla s = \lambda \left[\mathbf{w} - \rho J^{-1} \int_D (\mathbf{r} \times \mathbf{w})\, dm \times \mathbf{r}\right], \qquad \operatorname{div} \mathbf{w} = 0, \ \mathbf{w}|_\Gamma = 0 \tag{2.13}$$

Let us multiply the first equation of system (2.13) by $\overline{\mathbf{w}}$ (the bar denotes the complex conjugate) and integrate over the domain D, bearing in mind the fact that

$$\int_D \overline{\mathbf{w}} \Delta \mathbf{w}\, dm = -\int_D |\operatorname{rot} \mathbf{w}|^2\, dm, \quad \int_D \overline{\mathbf{w}} \nabla s\, dm = 0$$

$$\int_D \overline{\mathbf{w}} \left[\rho J^{-1} \int_D (\mathbf{r} \times \mathbf{w})\, dm \times \mathbf{r}\right] dm = \rho J^{-1} \int_D (\mathbf{r} \times \mathbf{w})\, dm \int_D (\mathbf{r} \times \overline{\mathbf{w}})\, dm \tag{2.14}$$

This yields

$$-\int_D |\operatorname{rot} \mathbf{w}|^2\, dm = \lambda \left[\int_D |\mathbf{w}|^2\, dm - \int_D (\mathbf{r} \times \overline{\mathbf{w}})\, dm \rho J^{-1} \int_D (\mathbf{r} \times \mathbf{w})\, dm\right] \tag{2.15}$$

Let us show that

$$F(\mathbf{w}) = \int_D |\mathbf{w}|^2 dm - \int_D (\mathbf{r}\times\overline{\mathbf{w}}) dm \rho J^{-1} \int_D (\mathbf{r}\times\mathbf{w}) dm > 0 \tag{2.16}$$

for $\mathbf{w} \neq 0$.

In fact

$$F(\mathbf{w}) = \int_D \left[\overline{\mathbf{w}} - \rho J^{-1}\int_D (\mathbf{r}\times\overline{\mathbf{w}})\, dm \times \mathbf{r})\right]\left[\mathbf{w} - \rho J^{-1}\int_D (\mathbf{r}\times\mathbf{w})\, dm \times \mathbf{r}\right] dm +$$
$$+ \int_D (\mathbf{r}\times\overline{\mathbf{w}})\, dm \rho J^{-1}\int_D (\mathbf{r}\times\mathbf{w})\, dm - \int_D \left[\rho J^{-1}\int_D (\mathbf{r}\times\overline{\mathbf{w}})\, dm \times \mathbf{r}\right] \times$$
$$\times \left[\rho J^{-1}\int_D (\mathbf{r}\times\mathbf{w})\, dm \times \mathbf{r}\right] dm \tag{2.17}$$

But

$$\int_D \left[\rho J^{-1}\int_D (\mathbf{r}\times\overline{\mathbf{w}})\, dm \times \mathbf{r}\right]\left[\rho J^{-1}\int_D (\mathbf{r}\times\mathbf{w})\, dm \times \mathbf{r}\right] dm =$$
$$= \left[\rho J^{-1}\int_D (\mathbf{r}\times\overline{\mathbf{w}})\, dm\right] \rho^{-1} J_1 \left[\rho J^{-1}\int_D (\mathbf{r}\times\mathbf{w})\, dm\right] = \int_D (\mathbf{r}\times\mathbf{w})\, dm \rho J^{-1} \times$$
$$\times \int_D (\mathbf{r}\times\overline{\mathbf{w}})\, dm - \left[\rho J^{-1}\int_D (\mathbf{r}\times\overline{\mathbf{w}})\, dm\right] \rho^{-1} J_0 \left[\rho J^{-1}\int_D (\mathbf{r}\times\mathbf{w})\, dm\right] \tag{2.18}$$

so that for $\mathbf{w} \neq 0$ we have

$$F(\mathbf{w}) = \int_D \left[\overline{\mathbf{w}} - \rho J^{-1}\int_D (\mathbf{r}\times\overline{\mathbf{w}})\, dm \times \mathbf{r}\right]\left[\mathbf{w} - \rho J^{-1}\int_D (\mathbf{r}\times\mathbf{w})\, dm \times \mathbf{r}\right] dm +$$
$$+ \left[\rho J^{-1}\int_D (\mathbf{r}\times\overline{\mathbf{w}})\, dm\right] \rho^{-1} J_0 \left[\rho J^{-1}\int_D (\mathbf{r}\times\mathbf{w})\, dm\right] > 0 \tag{2.19}$$

In the case of a smooth boundary Γ of the domain D we have [4]

$$\int_D |\operatorname{rot} \mathbf{w}|^2 dm \geqslant c \int_D |\mathbf{w}|^2 dm = c\|\mathbf{w}\|^2 \tag{2.20}$$

where the constant c depends on the domain D only. Hence,

$$|\lambda| \geqslant \int_D |\operatorname{rot} \mathbf{w}|^2 dm \frac{1}{\|\mathbf{w}\|^2} \geqslant c\frac{\|\mathbf{w}\|^2}{\|\mathbf{w}\|^2} = c > 0 \tag{2.21}$$

The author of [5] proves an existence theorem and investigates the properties of the spectrum of the solution of system (2.11). It should be noted, however, that in proving a relation of the type (2.16) he imposes a certain additional condition on the moments of inertia of the body.

Recalling relations (2.16) and (2.21), we infer from the results of [5] that for any initial distribution of the velocities $\mathbf{u}^\circ(\mathbf{r}, 0) \in W_2^1$ (e.g. see [2]) we have

$$\|\mathbf{u}(\mathbf{r}, \tau)\| \to 0, \quad \|\mathbf{u}(\mathbf{r}, \tau)\| < ae^{-\alpha\tau} \quad (|\lambda| > \alpha \geqslant c > 0) \tag{2.22}$$

as $\tau \to \infty$.

Here the norm is to be interpreted as the norm in $L_2(D)$. As $\tau \to \infty$ we have

$$\omega(\tau = \infty) = \omega^\circ + \rho J^{-1}\int_D (\mathbf{r}\times\mathbf{u}^\circ)\, dm \tag{2.23}$$

Let us formulate several lemmas, as in [3]. (A quantity satisfying the inequality $\|\cdot\| < a\xi$ will henceforth be denoted by $[[\xi]]$.)

L e m m a 2.1. The relations

$$\mathbf{u}_k^2(\mathbf{r}, \tau) = [[\tau^k]], \qquad \omega_k^2(\tau) = [[\tau^k]] \tag{2.24}$$

are valid.

Let us consider the systems in variations which result on substitution of series (2.2) into system (1.1) into which the "time" τ has been introduced.

The zeroth-approximation system coincides with (2.9), so that

$$\mathbf{u}_0^2(\mathbf{r}, \tau) = [[e^{-\alpha\tau}]] = [[1]], \qquad \omega_0^2(\tau) = [[1]] \tag{2.25}$$

Let us write out the first-approximation system

$$\frac{\partial \mathbf{u}_1^2}{\partial \tau} + \frac{d\omega_1^2}{d\tau} \times \mathbf{r} - \Delta \mathbf{u}_1^2 + \nabla q_1^2 = -[(\mathbf{u}_0^2 \nabla)\mathbf{u}_0^2 + 2\omega_0^2 \times \mathbf{u}_0^2], \ \operatorname{div} \mathbf{u}_1^2 = 0$$

$$J\frac{d\omega_1^2}{d\tau} + \frac{d\mathbf{K}_1^2}{d\tau} = -[\omega_0^2 \times (J\omega_0^2 + \mathbf{K}_0^2) - \mathbf{M}_0], \qquad \mathbf{K}_1^2 = \rho \int\limits_D (\mathbf{r} \times \mathbf{u}_1^2)\, dm \tag{2.26}$$

$$\mathbf{u}_1^2|_\Gamma = 0, \quad \mathbf{u}_1^2(\mathbf{r}, 0) = 0, \quad \omega_1^2(0) = 0$$

The eigenvalues of the homogeneous boundary value problem corresponding to system (2.26) are all real and negative ($|\lambda| > \alpha \geqslant c > 0$); it is clear, moreover, that the perturbations do not exceed some constant. Hence,

$$\mathbf{u}_1^2(\mathbf{r}, \tau) = [[\tau]], \qquad \omega_1^2(\tau) = [[\tau]] \tag{2.27}$$

The required result for the k-th approximation system is readily obtainable by mathematical induction.

Next, we note that the definitions of the functions $\mathbf{u}_k^3(\mathbf{r}, \tau)$, $\omega_k^3(\tau)$ themselves imply that

$$\mathbf{u}_k^3(\mathbf{r}, \tau) = [[\tau^k]], \qquad \omega_k^3(\tau) = [[\tau^k]] \qquad (k = 1, 2, 3, \ldots) \tag{2.28}$$

Let us introduce the functions

$$\pi_k(\mathbf{u}) = \mathbf{u}_k^2(\mathbf{r}, \tau) - \mathbf{u}_k^3(\mathbf{r}, \tau), \qquad \pi_k(q) = q_k^2(\mathbf{r}, \tau) - q_k^3(\mathbf{r}, \tau) \tag{2.29}$$

$$\pi_k(\omega) = \omega_k^2(\tau) - \omega_k^3(\tau) \qquad (k = 0, 1, 2, \ldots)$$

Lemma 2.2. If the initial conditions for the systems of equations in variations obtained on substituting series (2.1) into system (1.1) are specified in the way described below, then

$$\pi_k(\mathbf{u}) = [[e^{-\alpha\tau}]], \qquad \pi_k(\omega) = [[e^{-\alpha\tau}]] \quad (\alpha > 0) \tag{2.30}$$

Let us consider the systems of equations for $\pi_k(\mathbf{u})$, $\pi_k(q)$, $\pi_k(\omega)$ obtainable by subtracting from the equations for $\mathbf{u}_k^2(\mathbf{r}, \tau)$, $q_k^2(\mathbf{r}, \tau)$, $\omega_k^2(\tau)$ the analogous equations for $\mathbf{u}_k^3(\mathbf{r}, \tau)$, $q_k^3(\mathbf{r}, \tau)$, $\omega_k^3(\tau)$. The system of equations for $\pi_0(\mathbf{u})$, $\pi_0(q)$, $\pi_0(\omega)$ is of the same form as system (2.2), so that for any initial conditions we have

$$\pi_0(\mathbf{u}) = [[e^{-\alpha\tau}]] \tag{2.31}$$

Furthermore,

$$\pi_0(\omega) = -J^{-1}\pi_0(\mathbf{K}) + \pi_0(\omega)|_{\tau=0} + J^{-1}\pi_0(\mathbf{K})|_{\tau=0} \tag{2.32}$$

$$\pi_0(\mathbf{K}) = \rho \int\limits_D (\mathbf{r} \times \pi_0(\mathbf{u}))\, dm = [[e^{-\alpha\tau}]]$$

Let us stipulate that

$$\pi_0(\omega)|_{\tau=0} = -J^{-1}\pi_0(\mathbf{K})|_{\tau=0} \tag{2.33}$$

Under this condition we have the relation

$$\pi_0(\omega) = [[e^{-\alpha\tau}]] \tag{2.34}$$

Let us consider condition (2.33). By definition,

$$\pi_0(\omega)|_{\tau=0} = \omega_0^2(0) - \omega_0^3(0), \quad \pi_0(\mathbf{K})|_{\tau=0} = \rho \int_D [\mathbf{r}\times(\mathbf{u}_0^2(\mathbf{r},0) - \mathbf{u}_0^3(\mathbf{r},0)]\,dm$$

But

$$\omega_0^2(0) = \omega^\circ, \quad \mathbf{u}_0^2(\mathbf{r},0) = \mathbf{u}^\circ(\mathbf{r}), \quad \omega_0^3(0) = \omega_{00}^1 = \omega_0^1(0), \quad \mathbf{u}_0^3(\mathbf{r},0) = \mathbf{u}_{00}^1(\mathbf{r}) \quad (2.35)$$

For $\mathbf{u}^1_{00}(\mathbf{r})$ we have the system of equations

$$\Delta\mathbf{u}_{00}^1 - \nabla q_{00}^1, \quad \operatorname{div}\mathbf{u}_{00}^1 = 0, \quad \mathbf{u}_{00}^1|_\Gamma = 0 \tag{2.36}$$

Hence, $\mathbf{u}^1_{00} = 0$ and condition (2.33) is equivalent to the relation

$$\omega_0^1(0) = \omega_{00}^1 = \omega_0{}^\circ + \rho J^{-1}\int_D (\mathbf{r}\times\mathbf{u}^\circ(\mathbf{r}))\,dm \tag{2.37}$$

Now let us write out the system of equations for $\pi_1(\mathbf{u})$, $\pi_1(q)$, $\pi_1(\omega)$

$$\frac{\partial\pi_1(\mathbf{u})}{\partial\tau} + \frac{\partial\pi_1(\omega)}{d\tau}\times\mathbf{r} - \Delta\pi_1(\omega) + \nabla\pi_1(q) =$$
$$= -[(\mathbf{u}_0^2\nabla)\mathbf{u}_0^2 + 2\omega_0^2\times\mathbf{u}_0^2] + [(\mathbf{u}_0^3\nabla)\mathbf{u}_0^3 + 2\omega_0^3\times\mathbf{u}_0^3]$$

$$\operatorname{div}\pi_1(\mathbf{u}) = 0, \quad \pi_1(\mathbf{u})|_\Gamma = 0, \quad \pi_1(\mathbf{K}) = \rho\int_D (\mathbf{r}\times\pi_1(\mathbf{u}))\,dm \tag{2.38}$$

$$J\frac{d\pi_1(\omega)}{d\tau} + \frac{d\pi_1(\mathbf{K})}{d\tau} = -[\omega_0^2\times(\mathbf{K}_0^2 + J\omega_0^2)] + [\omega_0^3\times(\mathbf{K}_0^3 + J\omega_0^3)]$$

The eigenvalues of the homogeneous boundary value problem corresponding to system (2.38) are all real and negative ($|\lambda| > \alpha \geqslant c > 0$), and the perturbations clearly do not exceed the quantity $ae^{-\alpha\tau}$ in norm. Hence,

$$\pi_1(\mathbf{u}) = [[e^{-\alpha\tau}]] \tag{2.39}$$

Let us stipulate that

$$\pi_1(\omega)|_{\tau=0} = -J^{-1}\pi_1(\mathbf{K})|_{\tau=0} + J^{-1}\int_0^\infty \Phi\,d\tau \tag{2.40}$$

Here

$$\Phi = \omega_0^2\times(\mathbf{K}_0^2 + J\omega_0^2) - \omega_0^3\times(\mathbf{K}_0^3 + J\omega_0^3)$$

In this case

$$\pi_1(\omega) = -J^{-1}\pi_1(\mathbf{K}) + J^{-1}\int_\tau^\infty \Phi\,d\tau \tag{2.41}$$

But

$$\Phi = [[e^{-\alpha\tau}]], \quad \int_\tau^\infty \Phi\,d\tau = [[e^{-\alpha\tau}]], \quad \pi_1(\mathbf{K}) = [[e^{-\alpha\tau}]]$$

Hence, $\pi_1(\omega) = [[e^{-\alpha\tau}]]$, and, since $\pi_1(\mathbf{K}) = [[e^{-\alpha\tau}]]$, it follows that

$$\pi_1(\omega) = [[e^{-\alpha\tau}]] \tag{2.42}$$

Let us consider condition (2.40). By definition,

$$\pi_1(\omega)|_{\tau=0} = \omega_1^2(0) - \omega_1^3(0), \quad \pi_1(\mathbf{K})|_{\tau=0} = \rho\int_D [\mathbf{r}\times(\mathbf{u}_1^2(\mathbf{r},0) - \mathbf{u}_1^3(\mathbf{r},0))]\,dm$$

But

$$\omega_1^2(0) = 0, \quad \mathbf{u}_1^2(\mathbf{r},0) = 0, \quad \omega_1^3(0) = \omega_{10}^1 = \omega_1^1(0), \quad \mathbf{u}_1^3(\mathbf{r},0) = \mathbf{u}_{10}^1(\mathbf{r}) \tag{2.43}$$

and we have the following system of equations for finding $\mathbf{u}_{10}^1(\mathbf{r})$:

$$\Delta\mathbf{u}_{10}^1 - \nabla q_{10}^1 - \omega_{01}^1\times\mathbf{r} = 0, \quad \operatorname{div}\mathbf{u}_{10}^1 = 0$$
$$\mathbf{u}_{10}^1|_\Gamma = 0, \quad \omega_{01}^1 = J^{-1}[\mathbf{M}_0 - \omega_{00}^1\times J\omega_{00}^1] \tag{2.44}$$

Hence, condition (2.40) is one which is imposed on the initial data ω^1_{10} in the system obtained by substititing series (2.1) into system (1.1).

Imposing similar requirements on $\pi_k(\omega)$ for $\tau=0 (k=2,3,\ldots)$, we can use Lemma 2.1 to estimate the right sides and apply induction to obtain the required estimate for the k-th approximation system.

We note here that

$$\pi_k(\omega)|_{\tau=0} = -\omega_{k0}^1, \qquad \pi_k(\mathbf{u})|_{\tau=0} = -\mathbf{u}_{k0}^1(\mathbf{r}) \quad (k=2,3,\ldots)$$

and that the $\mathbf{u}_{k0}^1(\mathbf{r})$ can be found by solving the steady boundary value problems.

The same procedure can be used to verify the following relations which are a consequence of Lemma 2.1:

$$\frac{\partial}{\partial\tau}\pi_k(\mathbf{u}) = [[e^{-\alpha\tau}]], \qquad \frac{d}{d\tau}\pi_k(\omega) = [[e^{-\alpha\tau}]] \qquad (\alpha>0) \tag{2.45}$$

Lemma 2.3. In the interval $0 \leqslant t \leqslant -b\mu\ln\mu$, where b is some sufficiently large but fixed constant, we have

$$G_n(\mathbf{u}) - (\mathbf{u})_{n+1}^2 = [[\mu^{n+1}]], \qquad \frac{\partial}{\partial t}[G_n(\mathbf{u}) - (\mathbf{u})_{n+1}^2] = [[\mu^n]]$$

$$G_n(\omega) - (\omega)_{n+1}^2 = [[\mu^{n+1}]], \qquad \frac{d}{dt}[G_n(\omega) - (\omega)_{n+1}^2] = [[\mu^n]] \tag{2.46}$$

The proof of Lemma 2.3 is an exact repetition of the proof of the analogous lemma in [3].

Let us break up the segment $0 \leqslant t \leqslant T$ into two parts: the segment $0 \leqslant t \leqslant t^\circ = -b\mu\ln\mu$ and the segment $t^\circ \leqslant t \leqslant T$, where b is some sufficiently large constant which remains fixed as $\mu \to 0$.

Making use of Lemmas 2.2 and 2.3, we can show [3] the substitution of expressions (2.7) instead of the solutions into the first and the third equations of system (1.1) gives rise to the discrepancies $[[\mu^{n+1}]]$ and $[[\mu^{n+1}]] + [[\mu^n \exp(-\alpha t/\mu)]]$, respectively, in the segment $[0, t^\circ]$, and to the discrepencies $[[\mu^{n+1}]]$ in the segment $[t^\circ, T]$.

We note that by virtue of Lemma 2.2, in the segment $[t^\circ, T]$ we can take $(\mathbf{u})_n^1$, $(\omega)_n^1$ instead of $G_n(\mathbf{u})$, $G_n(\omega)$, respectively, in expressions (2.8). Let us introduce the functions

$$\begin{gathered} \mathbf{V}_{n+1} = \mathbf{u}(\mathbf{r}, t, \mu) - G_n(\mathbf{u}), \qquad \Omega_{n+1} = \omega(t, \mu) - G_n(\omega) \\ R_{n+1} = q(\mathbf{r}, t, \mu) - G_n(q) + \mu(\Omega_{n+1}\times\mathbf{r})(G_n(\omega)\times\mathbf{r}) \end{gathered} \tag{2.47}$$

These functions satisfy the following system of equations:

$$\mu\left[\frac{\partial}{\partial t}\mathbf{V}_{n+1} + \frac{d}{dt}\Omega_{n+1}\times\mathbf{r}\right] - \Delta\mathbf{V}_{n+1} + \nabla R_{n+1} + \\ + \mu[(\mathbf{V}_{n+1}\nabla)\mathbf{V}_{n+1} + 2\Omega_{n+1}\times\mathbf{V}_{n+1}] = \mathbf{f}_1$$

$$\operatorname{div}\mathbf{V}_{n+1} = 0, \quad \mathbf{V}_{n+1}|_\Gamma = 0, \quad \mathbf{K}_{n+1} = \rho\int_D(\mathbf{r}\times\mathbf{V}_{n+1})\,dm \tag{2.48}$$

$$J\frac{d}{dt}\Omega_{n+1} + \frac{d}{dt}\mathbf{K}_{n+1} + \Omega_{n+1}\times[J\Omega_{n+1} + \mathbf{K}_{n+1}] = \mathbf{f}_2$$

Here

$$\mathbf{f}_1 = [[\mu^{n+1}]], \quad \mathbf{f}_2 = [[\mu^{n+1}]] + [[\mu^n\exp(-\alpha t\mu^{-1})]] \quad \text{for} \quad 0\leqslant t\leqslant t^\circ$$

$$\mathbf{f}_1 = [[\mu^{n+1}]], \quad \mathbf{f}_2 = [[\mu^{n+1}]] \quad \text{for} \quad t^\circ\leqslant t\leqslant T \tag{2.49}$$

Let us consider the segment $[0, t^\circ]$. At the initial point $\mathbf{V}_{n+1} = 0$, $\Omega_{n+1} = 0$, so that we can linearize system (2.48) in the neighborhood of the point $t = 0$,

$$\mu\left[\frac{\partial}{\partial t}\mathbf{V}_{n+1} + \frac{d}{dt}\Omega_{n+1}\times\mathbf{r}\right] - \Delta\mathbf{V}_{n+1} + \nabla R_{n+1} = \mathbf{f}_1, \quad \operatorname{div}\mathbf{V}_{n+1} = 0$$

$$J\frac{d}{dt}\Omega_{n+1} + \frac{d}{dt}\mathbf{K}_{n+1} = \mathbf{f}_2, \qquad \mathbf{K}_{n+1} = \rho\int\limits_D(\mathbf{r}\times\mathbf{V}_{n+1})\,dm \tag{2.50}$$

$$\mathbf{V}_{n+1}|_\Gamma = 0, \quad \mathbf{V}_{n+1}(\mathbf{r}, 0) = 0, \quad \Omega_{n+1}(0) = 0$$

As we have already shown, the eigenvalues of the boundary value problem corresponding to system (2.50) are negative $(|\lambda| > \alpha/\mu \geqslant c/\mu > 0)$. Hence,

$$\mathbf{V}_{n+1}(\mathbf{r}, t, \mu) = \int\limits_0^t \exp(-\alpha(t-\tau)\mu^{-1})[[\mu^n]]\,d\tau = [[\mu^{n+1}]] \tag{2.51}$$

and, since

$$J\Omega_{n+1} + \mathbf{K}_{n+1} = \int\limits_0^t\{[[\mu^{n+1}]] + [[\mu^n\exp(-\alpha t\mu^{-1})]]\}\,dt = [[\mu^{n+1}]] \tag{2.52}$$

it follows that

$$\Omega_{n+1}(t, \mu) = [[\mu^{n+1}]] \tag{2.53}$$

By choosing μ sufficiently small, we can ensure the linearizability of system (2.48) over the entire segment $[0, t^\circ]$. We note that

$$\mathbf{V}_{n+1}|_{t=t^\circ} = [[\mu^{n+1}]], \quad \Omega_{n+1}|_{t=t^\circ} = [[\mu^{n+1}]]$$

Hence, choosing our μ sufficiently small, we find that

$$\mathbf{V}_{n+1}(\mathbf{r}, t, \mu) = [[\mu^{n+1}]], \quad \Omega_{n+1}(t, \mu) = [[\mu^{n+1}]] \quad \text{on} \quad [t^\circ, T] \tag{2.54}$$

as on $[0, t^\circ]$.

Relations (2.54), which are equivalent to inequalities (2.8), are therefore valid for $t \in [0, T]$ when μ is sufficiently small.

3. Using the above algorithm to construct an asymptotic form of the solution of problem (1.1) accurate to within terms of order μ^2 outside the boundary layer with respect to t. The system of equations for determining the zeroth approximation of the solution of problem (1.1) outside the time interval $[0, -b\mu\ln\mu]$ is of the form

$$J\frac{d\omega_0^1}{dt} + \omega_0^1\times J\omega_0^1 - \mathbf{M}(t) = 0, \quad [\mathbf{u}_0^1 = 0 \tag{3.1}$$

$$\omega_0^1(0) = \omega^\circ + \rho J^{-1}\int\limits_D(\mathbf{r}\times\mathbf{u}^\circ)\,dm$$

Assuming that the solution of system (3.1) has been found,

$$\omega_0^1(t) = \vartheta(t) \tag{3.2}$$

we can write out the system for determining the first approximation,

$$\Delta\mathbf{u}_1^1 - \nabla q_1^1 - \frac{d\vartheta(t)}{dt}\times\mathbf{r} = 0, \qquad \operatorname{div}\mathbf{u}_1^1 = 0$$

$$J\frac{d\omega_1^1}{dt} + \vartheta(t)\times J\omega_1^1 - J\vartheta(t)\times\omega_1^1 + \frac{d\mathbf{K}_1^1}{dt} + \vartheta(t)\times\mathbf{K}_1^1 = 0 \tag{3.3}$$

$$\mathbf{K}_1^1 = \rho \int_D (\mathbf{r} \times \mathbf{u}_1^1)\, dm, \qquad \mathbf{u}_1^1|_\Gamma = 0 \qquad \text{(cont.)}$$

$$\boldsymbol{\omega}_1^1(0) = -J^{-1}\left\{\rho \int_D (\mathbf{r} \times \mathbf{u}_1^1(r, 0))\, dm + \int_0^\infty [\boldsymbol{\omega}_0^2 \times (J\boldsymbol{\omega}_0^2 + \mathbf{K}_0^2) - \boldsymbol{\omega}_0^1(0) \times J\boldsymbol{\omega}_0^1(0)]\, d\tau\right\}$$

The values of $\boldsymbol{\omega}_0^2(\tau)$, $\mathbf{K}_0^2(\tau)$ can be determined from the system of equations

$$\frac{\partial \mathbf{u}_0^2}{\partial \tau} + \frac{d\boldsymbol{\omega}_0^2}{\partial \tau} \times \mathbf{r} - \Delta \mathbf{u}_0^2 + \nabla q_0^2 = 0, \qquad \operatorname{div} \mathbf{u}_0^2 = 0$$

$$J\,\frac{d\boldsymbol{\omega}_0^2}{d\tau} + \frac{d\mathbf{K}_0^2}{d\tau} = 0, \qquad \mathbf{K}_0^2 = \rho \int_D (\mathbf{r} \times \mathbf{u}_0^2)\, dm \qquad (3.4)$$

$$\mathbf{u}_0^2|_\Gamma = 0, \qquad \mathbf{u}_0^2(\mathbf{r}, 0) = \mathbf{u}^\circ(\mathbf{r}), \qquad \boldsymbol{\omega}_0^2(0) = \boldsymbol{\omega}^\circ$$

The solution of the steady boundary value problem in system (3.3) was constructed by Chernous'ko [1], whose results imply that

$$\rho \int_D [\mathbf{r} \times \mathbf{u}_1^1(\mathbf{r}, t)]\, dm = -\rho P\,\frac{d\boldsymbol{\vartheta}(t)}{dt} \qquad (3.5)$$

where P is a symmetric tensor dependent on the domain D only.

We therefore have the following system of equations for the function $\boldsymbol{\omega}_1^1(t)$

$$J\,\frac{d\boldsymbol{\omega}_1^1}{dt} + \boldsymbol{\vartheta}(t) \times J\boldsymbol{\omega}_1^1 - J\boldsymbol{\vartheta}(t) \times \boldsymbol{\omega}_1^1 - \rho\left[P\,\frac{d^2\boldsymbol{\vartheta}(t)}{dt^2} + \boldsymbol{\vartheta}(t) \times P\,\frac{d\boldsymbol{\vartheta}(t)}{dt}\right] = 0$$

$$\boldsymbol{\omega}_1^1(0) = J^{-1}\left\{\rho P\,\frac{d\boldsymbol{\vartheta}(t)}{dt}\bigg|_{t=0} - J\boldsymbol{\omega}_0^1(0) \times \int_0^\infty [\boldsymbol{\omega}_0^2(\tau) - \boldsymbol{\omega}_0^1(0)]\, d\tau\right\} \qquad (3.6)$$

The sum $\boldsymbol{\omega}_0^1(t) + \mu\boldsymbol{\omega}_1^1(t)$ represents the first two terms of the Maclaurin expansion of the solution of system (1.1) outside the boundary layer. It coincides with the solution of system (5.4) of [1] to within terms of the order μ^2 if the initial conditions for the latter system are given in the form $\boldsymbol{\omega}_0^1(0) + \mu\boldsymbol{\omega}_1^1(0)$.

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